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Optimisation of the potential shifting in the Martinelli–Parisi expansion of the $Z(2)$ gauge theory on a cubic lattice

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Abstract. As the Martinelli-Parisi expansion of recursive relations of the real space renormalisation group is not uniquely defined we test a recently proposed criterion for choosing among the different possibilities. We deal with the $Z(2)$ lattice gauge theory on a cubic lattice. Many alternatives are considered; all of these agree in the evaluation of the critical coupling constant and in excluding a second relevant operator at the non-trivial fixed point.

1. Introduction

Recently one of us computed the first order correction in the Martinelli-Parisi expansion [1] for the simplest non-trivial lattice gauge theory, i.e. for the three-dimensional $Z(2)$ model [2]. A significant improvement in the estimate of the critical coupling constant has been achieved. On the other hand, the Migdal-Kadanoff (MK) fixed point [3] acquires a new relevant direction and no longer governs the critical behaviour of the model with the Wilson action.

In [4] it has been shown that the Martinelli-Parisi expansion is not uniquely defined and that its predictions are strongly dependent on a choice of the potential shifting term. A general criterion to remove this ambiguity has been used in the two-dimensional Potts model in which only one more coupling constant is generated at first order and gives evaluations of critical quantities in good agreement with known results.

Before applying this strategy to more interesting gauge systems, it seems to us worthwhile to see it at work in the already complex recursion relations of the three-dimensional $Z(2)$ case. Such complexity appears with the generation of many coupling constants whose contribution is obtained through a structured sequence of computations and will be described in § 2. Various possible generalisations of the way in which one can here solve the ambiguity in the expansion are also discussed. Section 3 presents the different results, while some comments appear in the conclusion (§ 4).

2. Recursion equations

We shall consider the model with the action

$$S = \beta \sum_{\langle ijkl \rangle} \sigma_i \sigma_j \sigma_k \sigma_l \tag{2.1}$$

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where the σ are $Z(2)$ variables defined on links over a cubic lattice and $\langle ijkl \rangle$ are the links on the contour of an elementary plaquette. β denotes the coupling constant. At first order of the Martinelli-Parisi expansion all possible interactions on an elementary cube will be generated. We shall write them as $(\beta, \varepsilon\gamma_1, \varepsilon\gamma_2, \varepsilon\gamma_3, \varepsilon\gamma_4)$, where ε is a parameter associated with the magnitude of the shifted potential. γ are coupling constants of the interactions with the three-dimensional configurations (see [2]).

If one formally develops

$$\beta = \beta_0 + \varepsilon\beta_1 + \varepsilon^2\beta_2 + \dots \tag{2.2}$$

the first order correction to the MK recursion equation can be written as [2]

$$\begin{aligned} \beta_1' &= b_0\beta_0 + A_{00}\beta_1 + \sum_j A_{0j}\gamma_j \\ \gamma_i' &= b_i\beta_0 + \sum_j A_{ij}\gamma_j \end{aligned} \tag{2.3}$$

where i and j run from 1 to 4. The numerical values of b_0, b_i, A_{00}, A_{0j} and A_{ij} at the MK fixed point are given in the appendix.

The ambiguity in the Martinelli-Parisi expansion discussed in [2] arises from possible choices of the coupling constants in front of the plaquette shifting terms. Each such coupling constant in the x, y and z direction dilatations can take a contribution from the $O(\varepsilon)$ coupling constants. In any case the requirement of simplicity has limited us to considering only those coupling constants present in the symmetric case. Within this restriction, there are still two possibilities. In the first case, one can replace the β in the first potential shifting term with the $\tilde{\beta}$ given by

$$\tilde{\beta} \equiv \beta + \varepsilon \sum_i x_i\gamma_i. \tag{2.4}$$

In this way, the strength of the inner plaquettes in the yz direction is

$$\beta - (1-\varepsilon)\left(\beta + \varepsilon \sum_i x_i\gamma_i\right) \simeq \varepsilon\left(\beta_0 - \sum_i x_i\gamma_i\right) + O(\varepsilon^2) \tag{2.5}$$

while that of the outer ones is

$$\beta + (1-\varepsilon)\left(\beta + \varepsilon \sum_i x_i\gamma_i\right) \simeq 2(\beta_0 + \varepsilon\beta_1) - \varepsilon\left(\beta_0 - \sum_i x_i\gamma_i\right). \tag{2.6}$$

Such a change amounts to formally replacing $\varepsilon\beta_0$ by $\varepsilon(\beta_0 - \Sigma_i x_i\gamma_i)$.

In the second case, one can introduce the effect of such a transformation only in the symmetric final equations (2.3). These two possibilities coincide in the case in which only one potential shifting is performed to get the recursion equations as in the simple case with a single new coupling constant of the 2D Potts model [4].

It has been proposed to remove this ambiguity by minimising the effect of the perturbative operators with the strength $\varepsilon\gamma_i$ to the renormalisation of the unperturbative coupling constant $\tilde{\beta}$ at the non-trivial fixed point. In this case, since many coupling constants γ_i are present, it is possible to minimise the effect of each of them, imposing the relations

$$\left.\frac{\partial\tilde{\beta}'(\tilde{\beta}, \{\gamma_j\})}{\partial\gamma_i}\right|_{\tilde{\beta}^*, \{\gamma_j^*\}} = 0 \tag{2.7}$$

where $\tilde{\beta}'$ is $\beta' + \varepsilon\Sigma_i x_i\gamma_i'$ and the primes denote the quantities on the rescaled cubic lattice. $\tilde{\beta}^*$ and $\{\gamma_i^*\}$ are the fixed points of the recursion equations.

On the other hand, introducing a single x parameter, one can choose to minimise simply the total contribution of the perturbative operators by

$$\sum_i \gamma_i^* \frac{\partial \tilde{\beta}'}{\partial \gamma_i}\bigg|_{\tilde{\beta}^*,\{\gamma_j^*\}} = 0. \tag{2.8}$$

All these possibilities will be analysed in the following section.

3. Results

Let us start from the case in which we simply replace β_0 with $\beta_0 - \varepsilon \sum_i x_i \gamma_i$ in equation (2.3). The relations (2.7) give rise to the conditions

$$-(b_0 + A_{00})x_i + A_{0i} - \sum_j x_j [b_j x_i - A_{ji}] = 0 \tag{3.1}$$

which are equivalent to the system of equations

$$(b_0 + A_{00} + K)x_i - \sum_j x_j A_{ji} = A_{0i}$$

$$K = \sum_j x_j b_j. \tag{3.2}$$

The first part of (3.2) has the solutions

$$x_i = \sum_l \left(\sum_n A_{0n} B_{nl}^{-1}\right) [B_{li}/(b_0 + A_{00} + K - \lambda_l)] \tag{3.3}$$

where the matrix B diagonalises the matrix A with eigenvalues $\{\lambda_i;\ i = 1\text{-}4\}$:

$$\sum_{ij} B_{lj} A_{ji} B_{ik}^{-1} = \lambda_l \delta_{lk}. \tag{3.4}$$

Inserting the solutions (3.3) into the second part of (3.2), we get a polynomial equation of fifth degree in K.

As a consequence of (2.7), the recursion equation for $\tilde{\beta}_1$ becomes

$$\tilde{\beta}_1' = (b_0 + K)\beta_0^* + A_{00}\tilde{\beta}_1. \tag{3.5}$$

The fixed point solution of this equation should be comparatively small with respect to β_0^*. This fact determines the region in which solutions for K must be researched. We numerically find only one solution at $K = 0.623$. The corresponding $\{x_i\}$ are given in the appendix as $x_i(\mathrm{I})$. We obtain the critical coupling constant $\beta_{1C}(\mathrm{I})$

$$\beta_{1C} = \tilde{\beta}_1^* = [(b_0 + K)\beta_0^*]/(1 - A_{00}) = -0.266. \tag{3.6}$$

As the matrix elements A_{ji} of (2.3) are now replaced by $A_{ji} - b_j x_i(\mathrm{I})$, the eigenvalues of the linearised renormalisation group equation at the zeroth order are changed. They are reported in the appendix as $\{\lambda_i(\mathrm{I})\}$. A remarkable feature is that there is no longer a relevant eigenvalue in contrast to the $x_i = 0$ case [2]. This means that the non-trivial fixed point $\tilde{\beta}^*$, $\{\gamma_i^*\}$ is now responsible for the critical behaviour of the model (2.1).

As we have observed, one can use a single x parameter by the criterion (2.8). We have tried the case

$$x_i = x \qquad \text{for all } i \tag{3.7}$$

which leads to

$$-(b_0 + A_{00})x \sum_i \gamma_i^* + \sum_i A_{0i}\gamma_i^* - x^2 \sum_j b_j \sum_i \gamma_i^* + x \sum_{ji} A_{ji}\gamma_i^* = 0. \tag{3.8}$$

The fixed point solution for the γ_i is given by

$$\gamma_i^* = C_i \beta_0^* / (1 + xC) \tag{3.9}$$

where C_i and C are

$$\begin{aligned} C_i &= \sum_l B_{il}^{-1} (1 - \lambda_l)^{-1} \left(\sum_j B_{lj} b_j \right) \\ C &= \sum_i C_i. \end{aligned} \tag{3.10}$$

The matrix B and the eigenvalues λ_l are the same as before (see (3.4)). Inserting (3.9) into (3.8), we arrive at the second order equation in x

$$x^2 C \sum_j b_j + x \left(\sum_j b_j + C(b_0 + A_{00} - 1) \right) - \sum_j A_{0j} C_j = 0. \tag{3.11}$$

Only one solution of (3.11), i.e. $x(\mathrm{II}) = 3.221$, gives a reasonable result when inserted in (3.6) where K is now $K = x(\mathrm{II}) \Sigma_j b_j = 0.640$, and we get the correction to the critical coupling constant

$$\beta_{1C}(\mathrm{II}) = -0.286. \tag{3.12}$$

The eigenvalues $\{\lambda_l(\mathrm{II})\}$ of the matrix $A_{ji} - x(\mathrm{II}) b_j$ are collected in the appendix. As in the previous case, all of them are irrelevant.

When we are concerned with the other possible approach suggested in § 2 in which the replacement (2.4) is introduced only in the first potential shifting term, the criterion (2.7) leads to a set of conditions similar to (3.1). But now since b_0 and $\{b_i\}$ can be written as $t_0 + s_0$ and $\{t_i + s_i\}$, where t_0 and $\{t_i\}$ are the contributions obtained by the first decimation, only the terms of the form $t\beta_0$ must be changed into $t(\beta_0 - \Sigma_i x_i \gamma_i)$ (see the appendix for t values). Therefore equation (3.3) gives the solutions $\{x_i(\mathrm{III})\}$ with $K = \Sigma_j x_j(\mathrm{III}) t_j = 0.269$. On the other hand, in (3.5) K will still be $K = \Sigma_j x_j(\mathrm{III}) b_j = 0.552$. As a consequence, the critical point is now

$$\beta_{1C}(\mathrm{III}) = -0.183. \tag{3.13}$$

The linearised matrix around the fixed point has the eigenvalues $\{\lambda_i(\mathrm{III})\}$ of the matrix $A_{ji} - t_j x_i(\mathrm{III})$ reported in the appendix.

Moreover, the criterion (2.8) leads to an equation similar to (3.8) with the replacement $(b_0, \{b_j\}) \to (t_0, \{t_j\})$. The fixed point solution for the γ_i are given by

$$\gamma_i^* = \frac{C_i + x(C_i C' - C_i' C)}{1 + xC'} \beta_0^* \tag{3.14}$$

where C_i and C are the same as in (3.10) and C_i' and C' are obtained from C_i and C with the change of the b into t as before. The equation for x becomes

$$x^2 C' \sum_j b_j + x \left(\sum_j b_j + C(t_0 + A_{00} - 1) - \sum_i A_{0i}(C_i C' - C_i' C) \right) - \sum_j A_{0j} C_j = 0. \tag{3.15}$$

Here also only one of the solutions is interesting. It is $x(\mathrm{IV}) = 2.852$, and we get

$$\beta_{1C}(\mathrm{IV}) = -0.200. \tag{3.16}$$

For the eigenvalues $\{\lambda_i(\mathrm{IV})\}$, see the appendix. These eigenvalues also stay irrelevant, as in the three previous cases. We observe that the eigenvalue λ_2 is always approximately the same. As concerns λ_3 and λ_4, the cases (III) and (IV) give the same result as that of the $x = 0$ case. By contrast, they do not look well defined in the cases (I) and (II), where they become complex in the former case and λ_3 is negative in the latter. In any case they are always very small.

4. Conclusions

The Martinelli-Parisi expansion should provide a systematic improvement to the Migdal-Kadanoff recursion equation for the real space renormalisation group. It should be of considerable relevance in producing accurate formulae for the lattice gauge theory. As the 3D $Z(2)$ case is already an intricate model from such a point of view we have chosen it to test a novel criterion which should indicate how to construct this expansion in order to obtain reliable extrapolations in the critical region.

The main idea is that one can minimise the formally perturbative terms trying to make the series convergent. Due to the complexity of the starting calculation there appear many ways to substantiate such an attempt. Our first result is that all these possibilities give rise to the same kind of results. Secondly, the explicit calculation of the critical coupling constant leads to the first order evaluations

$$\begin{aligned} &\beta_C(\mathrm{I}) = 0.871 \\ &\beta_C(\mathrm{II}) = 0.861 \\ &\beta_C(\mathrm{III}) = 0.913 \\ &\beta_C(\mathrm{IV}) = 0.904 \end{aligned} \tag{4.1}$$

where the critical value which is obtained by the duality argument is $\beta_C = 0.761$. These values are not as good as in the $x = 0$ case $\beta_C = 0.767$ [2], but are still an improvement with respect to the MK result $\beta_C = 1.0044$. But what is more important is that in contrast to the previously considered computations [2] the critical behaviour always stays regulated by the non-trivial fixed point of the recursive equations. In fact the result of [2], in which a second relevant direction appeared at the MK fixed point, might be interpreted by the analysis of [5] and [6] as a pathology of the expansion in the presence of a crossover from second to first order phase transition.

So it is remarkable that with our choices such a mechanism does not occur and that there is evidence for a second order phase transition in agreement with what is known. We therefore think that approximate recursion equations of this type can still play a role in the study of gauge theories.

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Appendix

We give the numerical values of b_0, b_i, A_{00}, A_{0i} and A_{ij} in the recursion equations (2.3);

$$\begin{pmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix} = \begin{pmatrix} -0.395 \\ 1.100 \times 10^{-2} \\ 8.863 \times 10^{-2} \\ 7.799 \times 10^{-2} \\ 2.119 \times 10^{-2} \end{pmatrix}$$

$(A_{00}, A_{01}, A_{02}, A_{03}, A_{04}) = (1.861, 1.225, 4.913, 1.694, 2.537)$

$$\{A_{ij}\} = \begin{pmatrix} 0.882 & 0.801 & 0.145 & 1.382 \\ 1.057 & 7.299 & 3.511 & 3.766 \\ 0.531 & 5.193 & 2.644 & 2.319 \\ 0.589 & 1.388 & 0.476 & 1.309 \end{pmatrix} \times 10^{-1}.$$

The t_0 and t_i appearing in the cases (III) and (IV) in § 3 are given by

$$\begin{pmatrix} t_0 \\ t_1 \\ t_2 \\ t_3 \\ t_4 \end{pmatrix} = \begin{pmatrix} 9.171 \\ 0.585 \\ 4.587 \\ 3.325 \\ 0.887 \end{pmatrix} \times 10^{-2}.$$

The following numerical values are of x_i and λ_i for the cases (I)-(IV):

	x_i(I)	x(II)	x_i(III)	x(IV)
x_1	0.981		0.881	
x_2	4.650	3.221	4.124	2.852
x_3	1.896		1.665	
x_4	2.484		2.199	

	λ_i(I)	λ_i(II)	λ_i(III)	λ_i(IV)	$\lambda_i(x=0)$ [2]
λ_1	0.433	0.472	0.796	0.799	1.065
λ_2	0.138	0.140	0.138	0.138	0.138
λ_3	$0.009 + 0.002i$	-0.043	0.009	0.008	0.009
λ_4	$0.009 - 0.002i$	0.004	0.001	0.001	0.001

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